

Dimensional Reduction for Generalized Poisson Brackets

Ciprian Sorin Acatrinei*

Smoluchowski Institute of Physics, Jagellonian University
Reymonta 4, 30-059, Cracow, Poland

March 21, 2007

Abstract

We discuss dimensional reduction for Hamiltonian systems which possess nonconstant Poisson brackets between pairs of coordinates and between pairs of momenta. The associated Jacobi identities imply that the dimensionally reduced brackets are always *constant*. Some examples are given alongside the general theory.

1 Introduction

The dynamics of a classical conservative system with n degrees of freedom is specified once its Hamiltonian H and symplectic two-form ω are given as functions of the phase space variables $q_i, p_i, i = 1, \dots, n$. In most situations one mainly discusses specific Hamiltonians, assuming the symplectic form to be in canonical (constant and skew-diagonal) form

$$\omega_0 = \sum_{i,j=1}^n dq_i \wedge dp_j. \quad (1)$$

In a context to be specified below, we will study instead the influence on the dynamics of a generic symplectic form

$$\omega = \sum_{a,b=1}^{2n} \omega^{ab}(x) dx_a \wedge dx_b. \quad (2)$$

*On leave from: *National Institute of Nuclear Physics and Engineering - P.O. Box MG-6, 077125 Bucharest, Romania*; e-mail: acatrine@th.if.uj.edu.pl

Throughout this work the generalized $2n$ coordinates and momenta $\{q_i, p_i\}$ are occasionally denoted by $\{x_a, a = 1, \dots, 2n\}$. Darboux's theorem ensures that one can bring the generic form (2) into the canonical form (1), at least locally. However, even when the associated phase-space transformations can be explicitly performed, the Hamiltonian becomes more complicated in the new variables. For our purposes it will be more advantageous to work with the form (2), as it allows a transparent implementation of the Jacobi constraints.

This paper studies the limit in which a nonconstant symplectic form ω becomes singular (and degenerate, taking the form $\omega'_0 \sim \sum_{i,j=1}^{n'} dq_i \wedge dp_i, n' < n$ in Darboux coordinates). In this case the effect of the symplectic structure is maximal, and interesting results can be obtained for generic Hamiltonians. Consider Θ_{ab} to be the inverse of the $2n \times 2n$ matrix ω^{ab} ; it provides the Poisson brackets of the theory, $\{x_a, x_b\} = \Theta_{ab}$. The singular limit to be studied is the one in which the determinant of Θ_{ab} goes to zero, $\det \Theta \rightarrow 0$. In this limit $\omega \equiv \Theta^{-1}$ becomes singular, and the system experiences a dimensional reduction, as not all the x_a 's have independent evolution anymore. The reduced system displays a regular symplectic form which, surprisingly, turns out to be *constant*. The present work is organized around the formulation and proof of the above statement. In this way we generalize an alternative to Peierls dimensional reduction [1, 2], alternative which was up to now discussed only for constant Θ_{ab} [3]. A discussion of the limits of validity of our results is also given. A more explicit approach to the singular limit concludes the paper. The analysis is performed at the classical level. It can in principle be extended to quantum mechanics if a practical ordering prescription for functions of the noncommuting operators is given.

2 Constant symplectic form

Let us first review the constant ω case [3]. Consider

$$\omega = \sum_{a,b=1}^{2n} \omega^{ab} dx_a \wedge dx_b, \quad (3)$$

with the antisymmetric matrix ω^{ab} having constant entries. In the classical theory the inverse of ω^{ab} , $\Theta_{ab} = (\omega^{-1})_{ab}$, generates the extended Poisson brackets $\{x_a, x_b\} = \Theta_{ab}$. Quantum mechanically one replaces the Poisson brackets with commutators - a straightforward operation in the constant Θ case. The formalism we discuss is applicable to any space dimensionality; we

stay in $(2+1)$ -dimensions for clarity. The action

$$S = \int dt \left(\frac{1}{2} \omega_{ab} x_a \dot{x}_b - H(x) \right), \quad x_{1,2,3,4} = q_1, q_2, p_1, p_2, \quad (4)$$

engenders the equations of motion

$$\dot{x}_a = \{x_a, H\} = \Theta_{ab} \frac{\partial H}{\partial x_b}, \quad \Theta_{ab} = (\omega^{-1})_{ab}, \quad a, b = 1, 2, 3, 4. \quad (5)$$

Above the Poisson brackets are defined by $\{A, B\} = \Theta_{ab} \partial_a A \partial_b B$, in particular $\{x_a, x_b\} = \Theta_{ab}$. If we choose the symplectic form to be

$$\Theta = \begin{pmatrix} 0 & \theta & 1 & 0 \\ -\theta & 0 & 0 & 1 \\ -1 & 0 & 0 & F \\ 0 & -1 & -F & 0 \end{pmatrix} \quad \text{or} \quad \omega = \frac{1}{1 - \theta F} \begin{pmatrix} 0 & -F & 1 & 0 \\ F & 0 & 0 & 1 \\ -1 & 0 & 0 & -\theta \\ 0 & -1 & \theta & 0 \end{pmatrix}, \quad (6)$$

the Poisson brackets are $\{q_i, p_j\} = \delta_{ij}$, $\{q_1, q_2\} = \theta$, $\{p_1, p_2\} = F$, and the phase-space equations of motion become

$$\dot{q}_i = \frac{\partial H}{\partial p_i} + \theta \epsilon_{ij} \frac{\partial H}{\partial q_j}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} + F \epsilon_{ij} \frac{\partial H}{\partial p_j}, \quad \epsilon_{12} = -\epsilon_{21} = 1. \quad (7)$$

Dimensional reduction occurs when $\theta F = 1$. Then, the number of dynamical degrees of freedom is halved, since the equations of motion imply

$$\dot{q}_1 = -\theta \dot{p}_2 \quad \text{and} \quad \dot{q}_2 = \theta \dot{p}_1. \quad (8)$$

The degeneracy is a consequence of Θ being singular. The four-dimensional phase space $\{q_1, q_2, p_1, p_2\}$ collapses to a bidimensional one, spanned for instance by the now canonically conjugated variables q_1 and q_2 . As θ was taken to be constant, one has the identifications

$$q_1 = -\theta p_2 + c_1, \quad q_2 = \theta p_1 + c_2. \quad (9)$$

Freedom in imposing the initial conditions is insured by the arbitrariness of c_1 and c_2 .

In $(2+1)$ -dimensions, if the original Hamiltonian is rotationally invariant, $H = H(p_1^2 + p_2^2, q_1^2 + q_2^2)$, the resulting system is not only integrable - as any one-dimensional system is - but also easily solvable. The new Hamiltonian reads

$$H = H \left[\theta^2 (q_1^2 + q_2^2), q_1^2 + q_2^2 \right], \quad \{q_1, q_2\} = \theta. \quad (10)$$

In holomorphic coordinates

$$a = \frac{1}{\sqrt{2\theta}}(q_1 + iq_2), \quad \bar{a} = \frac{1}{\sqrt{2\theta}}(q_1 - iq_2), \quad 2\theta(q_1^2 + q_2^2) = \bar{a}a, \quad (11)$$

the equations of motion read

$$\frac{\dot{a}}{a} = -\frac{\dot{\bar{a}}}{\bar{a}} = i\frac{dh_\theta}{dn}, \quad h_\theta(n) = h_\theta(\bar{a}a) \equiv H, \quad (12)$$

and are solved in terms of trigonometric functions. At the quantum level immediate solvability is clear, as the dimensionally reduced Hamiltonian is a function h_θ of the harmonic oscillator Hamiltonian; the spectrum is discrete, with energy levels given by $E_n = h_\theta[\theta \times (n + 1/2)]$. The simplicity of the quantum reduced Hamiltonian does not seem to be widely appreciated; in the recent literature it was used in [4].

3 Planar case

As already stated, we will consider in this paper the more general case in which Θ in Eqs. (5,6) is x -dependent. Consider an arbitrary Hamiltonian $H(p, q)$ in (2+1)-dimensions and the following fundamental Poisson brackets

$$\{q_1, q_2\} = \theta(q, p), \quad \{p_1, p_2\} = F(q, p), \quad \{q_i, p_j\} = \delta_{ij}. \quad (13)$$

The associated Jacobi identities read

$$\frac{\partial\theta}{\partial q_1} - F\frac{\partial\theta}{\partial p_2} = 0, \quad \frac{\partial F}{\partial p_2} - \theta\frac{\partial F}{\partial q_1} = 0, \quad (14)$$

$$\frac{\partial\theta}{\partial q_2} + F\frac{\partial\theta}{\partial p_1} = 0, \quad \frac{\partial F}{\partial p_1} + \theta\frac{\partial F}{\partial q_2} = 0. \quad (15)$$

The equations of motion are identical in form to the ones written in (5), although Θ is not constant anymore. They combine to yield:

$$\dot{q}_1 + \theta\dot{p}_2 = (1 - F\theta)\frac{\partial H}{\partial p_1}, \quad \dot{p}_2 + F\dot{q}_1 = -(1 - F\theta)\frac{\partial H}{\partial q_2}, \quad (16)$$

$$\dot{q}_2 - \theta\dot{p}_1 = (1 - F\theta)\frac{\partial H}{\partial p_2}, \quad \dot{p}_1 - F\dot{q}_2 = -(1 - F\theta)\frac{\partial H}{\partial q_1}. \quad (17)$$

The above equations get halved in number if

$$F\theta = 1, \quad (18)$$

becoming

$$\dot{q}_1 + \theta \dot{p}_2 = 0, \quad \dot{q}_2 - \theta \dot{p}_1 = 0. \quad (19)$$

A remarkable fact, initially noted in [5], is that one has the following

Lemma 1 If $F\theta = 1$ the Hamiltonian system described by an arbitrary $H(p, q)$, the brackets (13) and the associated constraints (14,15), does satisfy

$$\dot{\theta} = 0. \quad (20)$$

To prove the lemma one uses the Jacobi identities (14,15) in order to rewrite $\dot{\theta} = \frac{\partial \theta}{\partial q_i} \dot{q}_i + \frac{\partial \theta}{\partial p_i} \dot{p}_i$ as

$$\dot{\theta} = \frac{\partial \theta}{\partial p_2} (\dot{p}_2 + F \dot{q}_1) + \frac{\partial \theta}{\partial p_1} (-\dot{p}_1 + F \dot{q}_2) = 0. \quad (21)$$

The equality to zero above follows from $F\theta = 1$ and Eq. (19).

An equivalent proof uses the equivalence of half of Eqs. (14,15) to $\{F, q_i\} = 0$ and $\{\theta, p_j\} = 0$. If θ and F are functions of each other however, $\{F, q_i\} = 0$ implies $\{\theta, q_i\} = 0$, whereas $\{\theta, p_j\} = 0$ leads to $\{F, p_j\} = 0$. All the Poisson brackets of F (or θ) vanish and in consequence

$$\dot{F} = \{F, x_a\} \partial_{x_a} H = 0, \quad \dot{\theta} = \{\theta, x_a\} \partial_{x_a} H, \quad \forall H(p, q).$$

In brief, the requirement $\det \Theta = 0$ and the Jacobi constraints suffice to enforce Eq.(20), even for a nonconstant symplectic form. The equations of motion were used in the proof, though for arbitrary Hamiltonian. Eqs. (19) and (20) permit again the identification (9), showing explicitly that only two out of the four initial x_a are independent - dimensional reduction takes place.

To show that θ is actually a trivial constant, not an integral of the motion, the above proof can be refined by explicitly solving the Jacobi constraints. If $F\theta = 1$, their number gets halved, as Eqs. (14,15) reduce to

$$\theta \frac{\partial \theta}{\partial q_1} - \frac{\partial \theta}{\partial p_2} = 0, \quad \theta \frac{\partial \theta}{\partial q_2} + \frac{\partial \theta}{\partial p_1} = 0. \quad (22)$$

Using the method of characteristics for partial differential equations of first order, the general solution $\theta(q_1, p_2, p_1, p_2)$ of the system (22) is found to be given implicitly by

$$\theta = \phi(q_1 + \theta p_2, q_2 - \theta p_1), \quad (23)$$

ϕ being an arbitrary function of two independent variables. Since (19) and (20) imply again that

$$q_1 + \theta p_2 = c_1, \quad q_2 - \theta p_1 = c_2,$$

Eq. (23) transforms into

$$\theta = \phi(c_1, c_2). \quad (24)$$

Only those combinations of the x 's which are becoming pure constants when $\theta F \rightarrow 1$ can enter θ . Consequently, θ^{red} – the value of θ in the dimensionally reduced space – is a pure constant itself. We thus proved

Theorem 1 Any planar dynamical system characterized by the brackets (13) dimensionally reduces in the limit $F\theta \rightarrow 1$. The reduced system exhibits a *constant* bracket between its canonically conjugated variables. If those are chosen to be q_1 and q_2 , that constant bracket is θ^{red} .

As a corollary, the solvability analysis presented at the end of Section 2 remains valid for the reduced system.

We present a second proof of this theorem, from the point of view of the reduced system; it does not use the equations of motion at all. θ^{red} is a constant if its total variation with respect to any initial phase space coordinate is zero, *after* dimensional reduction. Consider for instance variation with respect to q_1 . If we choose q_1, q_2 as independent (and canonically conjugated) coordinates of the reduced system, p_1 and p_2 will become functions of those. The *total* variation of θ^{red} with respect to q_1 is then

$$\frac{\delta\theta^{red}}{\delta q_1} = \frac{\partial\theta}{\partial q_1} + \frac{\partial\theta}{\partial p_i} \frac{\partial p_i}{\partial q_1}. \quad (25)$$

It will be independently proved later - Eq. (36) - that $\frac{\partial p_k}{\partial q^n} = F_{kn}$. Using this we obtain

$$\frac{\delta\theta^{red}}{\delta q_1} = \frac{\partial\theta}{\partial q_1} - F \frac{\partial\theta}{\partial p_2} = 0 \quad (26)$$

which vanishes due to Jacobi (14). The same can be shown with respect to q_2, p_1, p_2 . In checking this one obtains a very interesting interpretation of the Jacobi identities from the reduced space viewpoint: they *precisely* ensure that the total variation with respect to any variable is zero, cf. (26).

One can exemplify with a particular solution of the Jacobi constraints which exhibits dimensional reduction, for instance $\theta^{red} = 1/F^{red} = -\frac{q_1}{p_2}$. Then $\frac{\delta\theta^{red}}{\delta q_1} \sim p_2 - q_1 F^{red} = 0$. In conclusion, variation of any $x_a, a = 1, 2, 3, 4$, does not produce - thanks to Jacobi - a variation of θ^{red} , which is consequently a constant

A final consistency check can be performed: initially we started with relations of the type $\{q_1, p_2\} = 0$, which should remain true in the singular limit. Now, if $p_2 = -\frac{1}{\theta}q_1$, in the end $\{q_1, p_2\} \sim \{q_1, \theta\}$, which vanishes in the reduced case.

4 General result

We proceed to the case of an arbitrary number of dimensions. Consider first a generalized electromagnetic background $F(q, p)$, living on a space with noncommutativity field $\theta(q, p)$, and flat metric $g_{ij} = \delta_{ij}$:

$$\{q_i, q_j\} = \theta_{ij}(q, p) \quad \{q_i, p_j\} = \delta_{ij} \quad \{p_i, p_j\} = F_{ij}(q, p). \quad (27)$$

The quantum mechanical version is obtained by replacing the Poisson brackets with commutators, $\{, \} \rightarrow -i[,]$, and requires a practical prescription for operator ordering - this will not be addressed here.

A more elegant notation would use covariant and contravariant indices, writing q^k and θ^{ij} with the indices up. As no risk of confusion arises in this paper, we choose to keep all the indices down for simplicity.

The Jacobi identities read

$$\{q_k, F_{ij}\} = \frac{\partial F_{ij}}{\partial p_k} - \frac{\partial F_{ij}}{\partial q_m} \theta_{mk} = 0 \quad (28)$$

$$\{\theta_{ij}, p_k\} = \frac{\partial \theta_{ij}}{\partial q_k} + \frac{\partial \theta_{ij}}{\partial p_m} F_{mk} = 0 \quad (29)$$

$$\{F_{ij}, p_k\} + \text{cyclic} = \left(\frac{\partial F_{ij}}{\partial q_k} + \frac{\partial F_{ij}}{\partial p_m} F_{mk} \right) + \text{cyclic} = 0 \quad (30)$$

$$\{q_k, \theta_{ij}\} + \text{cyclic} = \left(\frac{\partial \theta_{ij}}{\partial p_k} - \frac{\partial \theta_{ij}}{\partial q_m} \theta_{mk} \right) + \text{cyclic} = 0. \quad (31)$$

They ensure the invariance of the commutation relations under time evolution, for a generic Hamiltonian $H(p, q)$. Explicitly:

$$\{\{q_m, p_n\} - \delta_{mn}, H\} = -\partial_{p_s} H \{F_{ns}, q_m\} + \partial_{q_s} H \{\theta_{ms}, p_n\}, \quad (32)$$

$$\{\{q_m, q_n\} - \theta_{mn}, H\} = -\partial_{p_s} H \{\theta_{mn}, p_s\} - \partial_{q_s} H (\{\theta_{ns}, q_m\} + \text{cyclic}), \quad (33)$$

$$\{\{p_m, p_n\} - F_{mn}, H\} = -\partial_{q_s} H \{F_{mn}, q_s\} + \partial_{p_s} H (\{F_{mn}, p_s\} + \text{cyclic}), \quad (34)$$

and Eqs. (28–31) ensure that the right hand side of Eqs. (32–33) is zero. The Jacobi identities above restrict F and θ to one of the following forms:

- F and θ both constant - the simplest situation, intensively studied recently under the name of noncommutative quantum mechanics [6]. The dimensional reduction has been worked out previously [3]
- θ constant, and $F(q, p)$ constrained by (28,30). This case is quite interesting for a variety of reasons. In particular, if $\theta_{ij} \neq 0$ then Eq. (28) forbids an electromagnetic field strength to depend only on the coordinates q . A detailed study appeared in [7]. Dimensional reduction cannot occur.

- F constant, and $\theta(q, p)$ constrained by (29,31) - the dual of the above.
- $F(q, p)$ and $\theta(q, p)$, constrained by (28–31) - the case of interest here.

We will first show in general that relations of the type " $F = \frac{1}{\theta}$ " arise *if and only if* the q 's and p 's are not independent, i.e. if dimensional reduction takes place. We work in an arbitrary number of dimensions, with the phase space spanned by the set $\{q_i, p_j\}$.

First, assume there exists a relation between the q_i 's and the p_j 's, say,

$$q_n = g_n(p_m), \quad p_m = f_m(q_s) = (g^{-1})_m(q_s). \quad (35)$$

Asking consistency of the commutation relations, one obtains equalities of the type $\{f(q(p)), q_s\} = \frac{\delta f}{\delta q_m} \theta_{ms} = -\frac{\delta f}{\delta q_m} \frac{\partial q_m}{\partial p_s}$. These imply

$$\frac{\partial q_m}{\partial p_s} = -\theta_{ms}, \quad \frac{\partial p_k}{\partial q_n} = F_{kn}. \quad (36)$$

Since $\frac{\partial q_m}{\partial p_k} \frac{\partial p_k}{\partial q_n} = \delta_{mn}$, it follows that

$$\theta_{mk} F_{kn} = -\delta_{mn}. \quad (37)$$

Eq. (37) actually amounts to $\det \Theta = 0$, if Θ provides the Poisson brackets (27). In two dimensions one immediately recovers the previous relation $F_{12} \equiv F = \theta^{-1} \equiv \theta_{12}^{-1}$. An alternative derivation uses the following chain of equalities,

$$\delta_{mn} = \{q_m, p_n\} = \{g_m(p), p_n\} = \frac{\partial q_m}{\partial p_s} F_{sn} = -\theta_{mk} F_{kn}. \quad (38)$$

Eq. (37) is consistent with the commutation relations, as illustrated by

$$\{q_m, q_n\} = \{g_m(p), g_n(p)\} = \frac{\partial g_m}{\partial p_s} F_{st} \frac{\partial g_n}{\partial p_t} = \theta_{ms} F_{st} \theta_{nt} = \theta_{mn}. \quad (39)$$

Conversely, we wish to prove that the constraint (37) implies dependence of the phase space variables (35). Eq. (37) and the equations of motion imply

$$\dot{q}_m + \theta_{mn} \dot{p}_n = \{q_m + \theta_{mn} p_n, H\} = 0, \quad \forall H(p, q). \quad (40)$$

Using (37), Eq. (40) can be put into the equivalent form

$$\dot{p}_m - F_{mn} \dot{q}_n = 0. \quad (41)$$

Eqs. (40,41) already show that dimensional reduction occurs, since the variations of the q 's and the p 's are related. To see it more explicitly, guided by

the work of the previous section, we calculate $\dot{\theta}_{ij}$. Using the Jacobi identities we have

$$\dot{\theta}_{ij} = \frac{\partial \theta_{ij}}{\partial x_a} \dot{x}_a = \frac{\partial \theta_{ij}}{\partial p_m} (\dot{p}_m - F_{mn} \dot{q}_n) = 0. \quad (42)$$

The last equality made use of (41). In consequence (the proof for \dot{F}_{ij} is identical) Eq. (37) and the Jacobi identities enforce

$$\dot{\theta}_{ij} = 0, \quad \dot{F}_{ij} = 0. \quad (43)$$

Eqs. (40,41) and (43) imply a precise relationship between the q 's and the p 's:

$$q_m = -\theta_{mn}(q, p)p_n + c_m, \quad \forall H(p, q), \quad (44)$$

which generalizes the two-dimensional relation (9).

One may enquire if (44) and (36) are compatible. Differentiating with respect to p one sees immediately that the necessary and sufficient conditions are precisely the Jacobi identities. Thus the whole set of constraints is consistent and we have demonstrated

Lemma 2 Given a system with $H(p, q)$ and Poisson brackets (27), the constraints (37) and (44) are equivalent. Either of them implies dimensional reduction, and consequently leads to (43).

$F(q, p)$ and $\theta(q, p)$ are thus constants of the motion. Since the analysis was independent of the form of the Hamiltonian, they are expected to be trivially constant. Otherwise, if for instance $H(p)$, then $\theta(q, p)$ would be a "constant of the motion" unrelated to the Hamiltonian. One way to demonstrate trivial constancy is to search for solutions of the Jacobi identities (28,29,30,31) and the constraint (37). One first notices an important fact: (37) ensures that Eqs. (30,31) are automatically satisfied, provided the first two identities, Eqs. (28,29), hold. The idea now is to show that the solutions of Eqs. (28,29) are given by functions which depend on $\bar{q}_m \equiv q_m + \theta_{mn}p_n$ (or equivalently on $\bar{p}_l \equiv p_l + F_{lm}q_m$) and to invoke (44) - which shows that exactly those combinations are constant. Let us sketch the proof of our claim. Consider $\theta_{ij} = f_{ij}(q_m + \theta_{mn}p_n) \equiv f_{ij}(\bar{q}_m)$. Then

$$\frac{\partial \theta_{ij}}{\partial q_k} = \frac{\partial f_{ij}}{\partial \bar{q}_m} (\delta_{km} + \frac{\partial \theta_{mn}}{\partial q_k} p_n), \quad \frac{\partial \theta_{ij}}{\partial p_m} = \frac{\partial f_{ij}}{\partial \bar{q}_s} (\frac{\partial \theta_{sn}}{\partial q_k} p_n + \theta_{sm}).$$

Denoting $\frac{\partial f_{ij}}{\partial \bar{q}_m} p_n$ by $A_{ij,mn}$ one sees that $\frac{\partial \theta_{ij}}{\partial q_k}$ and $\frac{\partial \theta_{ij}}{\partial p_m}$ are to be found from two systems of $n(n-1)/2$ linear equations each,

$$(A_{ij;sn} - \delta_{is}\delta_{jn}) \frac{\partial \theta_{sn}}{\partial q_k} = \frac{\partial f_{ij}}{\partial \bar{q}_k}, \quad (A_{ij;sn} - \delta_{is}\delta_{jn}) \frac{\partial \theta_{sn}}{\partial p_m} = \frac{\partial f_{ij}}{\partial \bar{q}_s} \theta_{sm}. \quad (45)$$

The two systems differ only through their inhomogeneous terms: the first set of inhomogeneous terms produces the second upon contraction with θ_{mn} . This immediately shows that the corresponding solutions of the two systems are obtained from one another through the same contraction - and this gives exactly the required Jacobi identity. Using the notation θ_{ij}^{red} , F_{ij}^{red} for the reduced system values, we have

Theorem 2 The Hamiltonian system described by an arbitrary $H(p, q)$ and by the Poisson brackets (27) dimensionally reduces if the constraint (37) is imposed. All the Poisson brackets are pure constants in the reduced system; in particular θ_{ij}^{red} and F_{ij}^{red} are all trivially constant.

It is instructive to give a second, simpler proof, from the point of view of the reduced system. It relies on one single but essential fact. The Jacobi identities, when restricted to the reduced system in which the x 's are related by (44), mean that the total variation of F^{red} or θ^{red} with respect to any coordinates is zero. That the Jacobi identity $\{\theta^{mn}, p_l\} = 0$ means zero total variation of θ in reduced space can be seen from the following sequence of equalities:

$$\{\theta_{mn}, p_l\} = \frac{\partial \theta_{mn}}{\partial q_l} + F_{sl} \frac{\partial \theta_{mn}}{\partial p_s} = F_{sl}^{red} \left(\frac{\partial \theta_{mn}^{red}}{\partial p_s} + \frac{\partial \theta_{mn}^{red}}{\partial q_r} \frac{\partial q_r}{\partial p_s} \right) = F_{sl}^{red} \frac{\delta \theta_{mn}^{red}}{\delta p_s} = \frac{\delta \theta_{mn}^{red}}{\delta q_l} \quad (46)$$

in which we used (36) repeatedly. Thus $\{\theta_{mn}, p_l\} = 0$ implies both $\frac{\delta \theta_{mn}^{red}}{\delta q_l} = 0$ and $\frac{\delta \theta_{mn}^{red}}{\delta p_s} = 0$. Similarly, $\{F_{mn}, q_l\} = 0$ implies $\frac{\delta F_{mn}^{red}}{\delta q_l} = F_{st}^{red} \frac{\delta F_{mn}^{red}}{\delta p_s} = 0$.

Thus F_{ij} and θ_{mn} are either constant, or display (thanks to Jacobi) only q, p -dependencies which lead to constancy when $q = q(p)$. In two-dimensions such examples are $F_{12}(p_1, q_2) = \theta_{12}^{-1} = \frac{p_1}{q_2}$; $F_{12}(p_2, q_1) = \theta_{12}^{-1} = -\frac{p_2}{q_1}$, or a more general one, $F_{12}(p_1, p_2, q_1, q_2) = \theta_{12}^{-1} = \frac{p_1 + p_2}{q_2 - q_1}$. It is easy to check that any of those satisfies $\{F_{12}, q_l\} = 0$, once $F_{12} = \theta_{12}^{-1}$.

We showed that for a large class of systems the Poisson brackets of the reduced system are not only time-independent, but completely constant. In doing so we considerably generalized an alternative [3] to the celebrated Peierls reduction [1, 2] in strong magnetic fields. An elementary corollary of our aproach is immediate integrability of the reduced system, at least if the starting point is a rotationally invariant planar Hamiltonian. Some higher dimensional examples are currently under investigation.

5 Other cases

Consider the most general planar Poisson brackets

$$\{q_1, q_2\} = \theta(q, p), \quad \{p_1, p_2\} = F(q, p), \quad \{q_i, p_j\} = g_{ij}(q, p) \quad (47)$$

and their analogues in higher dimensions. The brackets g_{ij} became generic functions of q_i, p_j . We would like to clarify in which conditions the results of Sections 3 and 4 continue to hold.

The equations of motion are of the form $\dot{x}_a = \Theta_{ab} \frac{\partial H}{\partial x_b}$ and are written down straightforwardly by use of (47). In $(2+1)$ - dimensions the null-determinant condition $\det \Theta = 0$ reads

$$\theta F - g_{11}g_{22} + g_{12}g_{21} = 0. \quad (48)$$

It permits to show that four linear combinations of the time derivatives of the x_a 's give zero

$$F\dot{q}_1 - g_{12}\dot{p}_1 + g_{11}\dot{p}_2 = 0, \quad F\dot{q}_2 - g_{22}\dot{p}_1 + g_{21}\dot{p}_2 = 0, \quad (49)$$

$$g_{22}\dot{q}_1 - g_{12}\dot{q}_2 + \theta\dot{p}_2 = 0, \quad g_{21}\dot{q}_1 - g_{11}\dot{q}_2 + \theta\dot{p}_1 = 0. \quad (50)$$

However only two of the above constraints are independent, showing that only two out of the four time derivatives $\dot{q}_1, \dot{q}_2, \dot{p}_1, \dot{p}_2$ have independent evolution. Dimensional reduction takes place when (48) is satisfied.

Define for future use the linear differential operators

$$D_1 = -\theta \frac{\partial}{\partial q_2} - g_{11} \frac{\partial}{\partial p_1} - g_{12} \frac{\partial}{\partial p_2}, \quad D_2 = -\theta \frac{\partial}{\partial q_1} + g_{21} \frac{\partial}{\partial p_1} + g_{22} \frac{\partial}{\partial p_2}, \quad (51)$$

$$D_3 = -F \frac{\partial}{\partial p_2} + g_{11} \frac{\partial}{\partial q_1} + g_{21} \frac{\partial}{\partial q_2}, \quad D_4 = +F \frac{\partial}{\partial p_1} + g_{12} \frac{\partial}{\partial q_1} + g_{22} \frac{\partial}{\partial q_2}. \quad (52)$$

Only two of them are linearly independent, as for instance

$$g_{11}D_2 = g_{12}D_3 - \sigma D_1, \quad -g_{11}D_4 = \theta D_3 + g_{21}D_1, \quad (53)$$

but using all four on equal footing simplifies the writing. To calculate a time derivative $\dot{f} = \frac{\partial f}{\partial x_c} \dot{x}_c$ of a function $f(x)$ we observe (after straightforward manipulation of the equations of motion) that

$$\frac{d}{dt} = \frac{\partial H}{\partial q_1} D_1 - \frac{\partial H}{\partial q_2} D_2 + \frac{\partial H}{\partial p_1} D_3 + \frac{\partial H}{\partial p_2} D_4. \quad (54)$$

To ensure that $\dot{f} = 0$, it is thus enough to show - given Eqs. (53) - that $D_1 f = 0$ and $D_3 f = 0$.

Additional information is given by the Jacobi identities, which read

$$D_3\theta + D_1g_{21} + D_2g_{11} = 0, \quad D_4\theta + D_1g_{22} + D_2g_{12} = 0, \quad (55)$$

$$D_1F - D_3g_{12} + D_4g_{11} = 0, \quad D_2F - D_3g_{22} + D_4g_{21} = 0. \quad (56)$$

(Of course, the differential operators are supposed to act on all the functions which are put on their right.) There is not enough information in (55,56) to show that D_1 and D_3 (say) annihilate any of the functions F , θ or g_{ij} . We must retreat to particular cases.

If the g_{ij} 's are constant one obtains $D_1\theta = D_4\theta = 0$, $D_2F = D_3F = 0$, enough to prove their constancy; F and θ are also related via (48).

However, if we take $g_{12} = g_{21} = 0$, but all the other four nonconstant, all that we can reach is

$$D_3\left(\frac{\theta}{g_{11}}\right) = D_3\left(\frac{g_{22}}{F}\right) = 0 \quad D_1\left(\frac{\theta}{g_{22}}\right) = D_1\left(\frac{g_{11}}{\theta}\right) = 0. \quad (57)$$

which is not enough to prove anything, except that $\frac{\theta F}{g_{11}g_{22}}$ is constant. However this expression is already known to be equal to 1, cf. (48).

An intermediate situation appears if three of the six functions present in (47) are nonconstant, e.g. θ , F and one g , say g_{12} . Then one can use (48) to express g_{12} as a function of θ and F , which are then constrained to be constant by the four identities (55,56).

We see that not only the Poisson structure (13) has the remarkable property of forcing its dimensionally reduced brackets to be constant. The condition $\det \Theta = 0$ allows in fact for one more nonconstant bracket in (47). Making use of the symmetry between the x 's, the three nonconstant brackets in (13) can be picked at will, and we have demonstrated

Theorem 3 If at most three out of the six functions appearing in (47) are nonconstant and the dimensional reduction condition (48) is imposed, then the reduced brackets are all constant.

It is now clear why the property of the structure (13) generalizes to any even number of dimensions, namely for brackets of the type (27). In this case one has $n(n-1)$ functions, θ_{ij} , F_{ij} , $i = 1, 2, \dots, n$, on which the Jacobi identities (28,29) impose exactly the same number of constraints, namely that each of them be annihilated by the operator $\frac{\partial}{\partial p_k} - \theta_{mk} \frac{\partial}{\partial q_m}$. Here too one single nonconstant g can be added to the game, as it is determined by θ_{ij} and F_{ij} via the $\det \Theta = 0$ condition. Using again the freedom to relabel the q 's and p 's at will, one can prove our most general statement:

Theorem 4 Given a $2n$ -dimensional phase space $\{x_a, a = 1, \dots, 2n\}$ with $2n^2 - n$ brackets $\Theta_{ab} = \{x_a, x_b\}$, if at most $n^2 - n + 1$ of them are given by nonconstant functions, then the condition $\det \Theta = 0$ forces all of them into constants.

6 The singular limit explicitly

In the previous sections the constraints posed by the Jacobi identities were solved in the dimensionally reduced (singular) case. To explicitly see dimensional reduction at work, we wish to solve in $(2+1)$ -dimensions the initial constraints (14,15). Since those can be completely separated in equations containing derivatives either with respect to q_1 and p_2 , or q_2 and p_1 , we consider only the pair (q_1, p_2) . Defining for simplicity $u = F$, $1/\theta = v$, $q_1 = x$, $p_2 = y$, $\partial_x = \partial_1$, $\partial_y = \partial_2$, the equations to be solved are:

$$\partial_1 u - v \partial_2 u = 0, \quad \partial_1 v - u \partial_2 v = 0. \quad (58)$$

We are interested in the existence of solutions with $u \neq v$ in general, but with $u \rightarrow v$ as a parameter is varied (a smooth approach to dimensional reduction in a sense). We will forget for a moment about the physical dimensions of q, p, F, θ , as these can be easily reinstated at a later stage through the introduction of dimensionfull parameters. Since the above system is nonlinear, but with coefficients not depending on the two independent variables, we use the hodograph method - we invert the roles of the dependent and independent variables. Seeing x and y as functions of u and v is possible if the Jacobian of the transformation

$$J = \partial_x u \partial_y v - \partial_y u \partial_x v$$

is nonzero (otherwise one easily shows that all solutions satisfy $u = v$). Eqs. (58) become

$$\partial_u y + u \partial_u x = 0, \quad \partial_v y + v \partial_v x = 0. \quad (59)$$

One shows easily that the only solutions with $u \neq v$ of the above system of linear partial differential equations are of the form

$$x(u, v) = f(u) + g(v), \quad y(u, v) = - \int u \frac{df}{du} - \int v \frac{dg}{dv}, \quad (60)$$

with f and g arbitrary functions of one variable. To fix f and g one needs boundary conditions or initial conditions coming from a physical criterion. We do not adress this here, but merely present a few mathematically simple choices for f and g .

Consider first the choice $f = \alpha u$, $g = -\alpha v$. Then one obtains in the end

$$u = -\frac{y}{x} + \frac{x}{2\alpha}, \quad v = -\frac{y}{x} - \frac{x}{2\alpha}. \quad (61)$$

Restoring dimensionality one sees that α has dimension $[length]^3$. It is immediately seen that $\lim_{\alpha \rightarrow \infty} u = \lim_{\alpha \rightarrow \infty} v = -y/x$ reproduces the reduced solution mentioned earlier.

Another simple choice, $f(u) = \alpha \log(u/u_0)$, $g(v) = -\alpha \log(v/u_0)$, leads to

$$u = \frac{(y/\alpha)e^{x/\alpha}}{1 - e^{x/\alpha}}, \quad v = \frac{(y/\alpha)}{1 - e^{x/\alpha}}. \quad (62)$$

This time α has the physical dimension of $[length]^1$ and $[u_0] = [length]^{-2}$. The limiting behaviour is the same as in the first example, $u = v = -\frac{y}{x}$ or, in initial notation, $F = 1/\theta = -\frac{p_2}{q_1}$.

An apparently innocent change of the above can make a big difference. The choice $f(u) = \alpha \log(u/u_0)$, $g(v) = \alpha \log(v/v_0)$, with α a length and u_0, v_0 having dimension $[length]^{-2}$, leads to

$$u = \frac{-y/\alpha + \sqrt{y^2/\alpha^2 - 4u_0v_0e^{x/\alpha}}}{2}, \quad v = \frac{-y/\alpha - \sqrt{y^2/\alpha^2 - 4u_0v_0e^{x/\alpha}}}{2},$$

or viceversa. This is an example of solution which does not display dimensional reduction ($u = v$) in any limit.

Acknowledgements

During its various stages this work was supported by the Marie Curie Actions Transfer of Knowledge Project COCOS (contract MTKD-CT-2004-517186), by NATO Grant PST.EAP.RIG.981202, and by Romanian Grant CEE-05-D11-49.

References

- [1] R. Peierls, *Z. Phys.* **80** (1933) 763.
- [2] G. Dunne and R. Jackiw, *Nucl. Phys. Proc. Suppl.* 33C (1993) 114; R. Jackiw, hep-th/9306075.
- [3] See for instance C.S. Acatrinei, *JHEP* 0109 (2001) 007; *Rom. J. Phys.* 52 (2007) 3, and references therein.
- [4] S. Bellucci, A. Nersessian, C. Sochichiu, *Phys. Lett.* B522 (2001) 345.
- [5] C.S. Acatrinei, *Rom. J. Phys.* 51 (2006) 343.
- [6] It is impossible to do justice to this topic here. See e.g. references in and citations of V.P. Nair and A.P. Polychronakos, *Phys. Lett.* B505 (2001) 267.
- [7] C.S. Acatrinei, *Mod. Phys. Lett.* A20 (2005) 1437; hep-th/0402049.